thruster floating potential rose steeply as keeper current was reduced. A keeper current of 0.45 amp was sufficient for most conditions in the test. With minor differences, the keeper voltage and thruster floating potential exhibited similar trends at 4250 and 6130 test hours. Reducing tip heater power tended to raise the keeper voltage and floating potential. It is believed that some minimum cathode temperature is necessary to preserve the low work function surface which enhances electron emission. A set of data points was taken at 7100 hr with the tip heater power increased to 10 w. As shown in the history (Fig. 1c and d), both the keeper voltage and the thruster floating potential decreased together. The long-term rise in neutralizer keeper voltage was accompanied by a similar rise in floating potential. Addition of up to 10 w of heater power lowered the keeper voltage and floating potential as shown at 6500 and again at 7000 hr of operation.

Thrust vectoring subsystem. Two types of thrust vectoring grids were used in the durability test. The translating grid has been reported extensively.[4,7] In the present test, after 2023 hr, grid erosion was minimal, giving an extrapolated life of over 20,000 hr.[8]

After starting the electrostatic vector test at 2023 thruster hours, the thruster was not exposed to atmosphere. The grid was operated in the beam deflection mode for a total of 1900 hr in four directions. About 120 hr were at a maximum deflection of 5° based on a deflection voltage of 250 v. The remaining hours were at deflection angles of 2°–4°. These hours approximate the total vectoring requirements of a proposed 5-yr satellite stationkeeping thruster.

Grid shorting noted in the time history occurred between grid elements as well as between the screen and accelerator electrodes. Most shorts were cleared by sustained application of 200–400 v at currents ranging from 6 ma to 70 ma. An alternate clearing method in which a capacitor was discharged across the grid short was highly successful. These shorts are believed to be caused by accumulation of sputtered metal which build up in the intersections, then peel off. Interelectrode shorts between the screen and accelerator may also be caused by buildup of sputtered metal or sputtered metal flakes shed from ion chamber surfaces.

Propellant feed system test

The purpose of the propellant reservoir and CIV assembly test was to evaluate the following factors: 1) long term retention capability of the propellant reservoir pressurizing gas; 2) vaporizer flow control over a long operating period; 3) propellant feed isolator durability under simulated thruster operation; and 4) cathode durability. By the nature of the test, all four factors were evaluated concurrently over the test duration of 5400 hr. The vaporizer was operated at a constant temperature of $340 \pm 2°C$, and 1300 v was impressed across the isolator.

The volume change of the gas, and hence the displacement of mercury propellant was linear with time. Thus the flow rate was constant over the test period and the gas reservoir did not leak. The flow rate of 30.4 ma calculated from the pressure data agreed favorably with those obtained by other means including mass measurements. In the thruster durability test, the same operating temperature on a vaporizer of identical design yielded a propellant flow of 33–34 ma.

The isolator was tested at 1300 v with the cathode operating at conditions representative of thruster operation. The leakage current throughout the test remained at less than 0.1 μa. Electrical breakdowns occurred across the isolator when the voltage was raised to 1350 v.

The evaluation of cathode durability in this test was not a rigorous in-thruster evaluation because of the differences in the configuration and the discharge plasma. Visual inspection after 5400 hr showed no discernible erosion of either the cathode tip or the keeper aperture. All components of the CIV assembly were in functional order, but the test was terminated to make way for a second generation CIV assembly designed to operate at 1600 v.

Conclusions

Durability tests of a structurally integrated 5-cm-diam ion thruster system have been conducted at the Lewis Research Center. The Hughes SIT-5 thruster modified for specific tests has operated over 8000 hr. A history of the test and performance mapping at widely separated intervals showed that the cathode, ion chamber discharge, and accelerator drain current characteristics remained essentially constant throughout the test. Restarts were reliable and easy even after 7900 hr of operation. The neutralizer keeper voltage and the thruster floating potential showed a slight rise with time. A translating screen thrust vector grid showed minimal erosion after 2023 hr. In situ examination of the electrostatic thrust vector grid showed some erosion due to charge exchange and possibly direct impingement ions. Grid shorting was a recurrent problem, but all shorts were cleared to permit continuation of the test. An independent test of the SIT-5 propellant feed system conducted for 5400 hr has demonstrated its reliability. The test was terminated to make way for a second generation cathode-isolator-vaporizer assembly designed to operate with 1600 v across the isolator.

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Forced Motion of Lumped Mass Systems Including the Effect of Axial Force

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Nomenclature

p = scalar parameter
$(p_{cr})_j$ = critical buckling loads $j = 1, \ldots, N$
$(p_{cr})_{min}$ = minimum critical buckling load

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ω_j	= natural frequency of free vibration, neglecting axial forces $j = 1, \ldots, N$
Ω_j	= natural frequency of free vibration, including axial forces $j = 1, \ldots, N$
$\{f(t)\}$	= column vector of time-dependent external forces
$\{x(t)\}$	= column vector of generalized displacements
$[C]$	= diagonal matrix with terms $\cos \Omega_j t, j = 1, \ldots, N$
$[I]$	= unit matrix
$[K]$, $[M]$, $[P]$	= stiffness, mass, and axial force matrices
$[[K] - p[P]]$	= effective stiffness matrix
$[S]$	= diagonal matrix with terms $\sin \Omega_j t$, $j = 1, \ldots, N$
$[\hat{S}]$	= diagonal matrix with terms $\sin \Omega_j(t - \tau)$, $j = 1, \ldots, N$
$[U]$, $[U_f]$, $[U_s]$	= eigenvector or modal matrices
$[\Lambda_{kp}]$	= diagonal form of the effective stiffness matrix
$[\Lambda_{kf}]$, $[\Lambda_{ks}]$	= diagonal forms of the stiffness matrix
$[\Lambda_m]$, $[\Lambda_{mf}]$	= diagonal forms of the mass matrix
$[\Lambda_p]$	= diagonal form of the axial force matrix
$[\Lambda_\omega]$	= diagonal matrix with terms equal to the square of the natural frequencies, neglecting axial forces
$[\Lambda_\Omega]$	= diagonal matrix with terms equal to the square of the natural frequencies, including axial forces

Introduction

THE vibration of undamped dynamical systems with lumped parameters has been recently studied. Both the free and forced vibration problem have been considered.[1,2] The addition of conservative external axial forces and their effect on the stiffness matrix have been discussed.[3]

The purpose of this Note is to extend this latter work concerning the addition of axial forces in much greater detail. A more formal compact matrix solution of the forced vibration problem is presented which eliminates the use of a series-type solution. This matrix formulation necessitates the computation of the inverse of diagonal matrices only, which is an extremely important consideration for computer time requirements of large scale-systems. More importantly, the mathematical conditions under which simultaneous diagonalization of the mass, stiffness, and the axial force matrices takes place are formulated.

Analysis

The equations of motion of an undamped N-degree of freedom dynamical system with lumped parameters including the effect of conservative axial forces are written in matrix form as

$$[M]\{\ddot{x}\} + [[K] - p[P]]\{x\} = \{f(t)\} \tag{1}$$

For simplicity, the rigid body motions are neglected. Hence, the mass, stiffness, the axial force matrices (i.e., $[M]$, $[K]$, and $[P]$) are positive definite and nonsingular. The parameter p is a scalar quantity.

The solution of Eq. (1) using modal analysis is written in Duhamel integral form as

$$\{x(t)\} = [U][C]\{a(0)\} + [U][S]\{b(0)\} + [U][\Lambda_\Omega^{-1/2}][\Lambda_m^{-1}]\int_{\tau=0}^{\tau=t} [\hat{S}][U^T]\{f(\tau)\}d\tau \tag{2a}$$

where

$$\{a(0)\} = [\Lambda_m^{-1}][U^T][M]\{x(0)\} \tag{2b}$$

and

$$\{b(0)\} = [\Lambda_\Omega^{-1/2}][\Lambda_m^{-1}][U^T][M]\{\dot{x}(0)\} \tag{2c}$$

The matrices $[C]$, $[S]$, and $[\hat{S}]$ are diagonal matrices with terms $\cos \Omega_j t$, $\sin \Omega_j t$, and $\sin \Omega_j(t-\tau)$, respectively, $j = 1, \ldots, N$. The terms Ω_j, $j = 1, \ldots, N$, are the natural frequencies of free vibration of the system including axial forces. The matrix $[\Lambda_\Omega]$ is a diagonal matrix containing terms Ω_j^2, $j = 1, \ldots, N$. The modal matrix $[U]$ simultaneously diagonalizes the mass matrix and the effective stiffness matrix in the form

$$[U^T][M][U] = [\Lambda_m] \text{ and } [U^T][[K] - p[P]][U] = [\Lambda_{kp}] \tag{3}$$

where

$$[\Lambda_{kp}] = [\Lambda_m][\Lambda_\Omega] \tag{4}$$

It should be noted carefully that the $[U]$ matrix does not diagonalize the $[K]$ and $[P]$ matrix individually. The $[U]$ matrix is orthogonal if and only if the matrices $[M]$ and $[[K] - p[P]]$ are commutative.[4]

For a given set of external forces $\{f(t)\}$ and a given set of initial conditions $\{x(0)\}$ and $\{\dot{x}(0)\}$, Eq. (2) yields after proper integration the dynamic response vector $\{x(t)\}$. The solutions of Eq. (2) are stable in the sense of small oscillations about the undeformed equilibrium position subject to two conditions. Firstly, the parameter p of Eq. (1) must be less than the least value of the critical buckling load $(p_{cr})_{min}$ obtained by setting the terms $\Omega_j = 0$, $j = 1, \ldots, N$, solving for the terms $(p_{cr})_j$, $j = 1, \ldots, N$, and extracting the smallest value. Secondly, the frequencies of the applied external forces must be different than any natural frequency of the system, Ω_j, $j = 1, \ldots\ n$.

The form of Eq. (2) requires the inversion of the diagonal mass and natural frequency matrices only, which is an important consideration for computer time requirements of large-scale systems.

Simultaneous Diagonalization

A special case arises when the same eigenvector matrix simultaneously diagonalizes the $[M]$, $[K]$, and $[P]$ matrices individually. The mathematical requirements for this condition are obtained by observing the solutions of the free vibration problem neglecting axial forces, and the solutions of the static stability problem.

a. Free vibration problem neglecting axial forces

Equation (1) is rewritten

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\} \tag{5}$$

Let $[\Lambda_\omega]$ be a diagonal matrix with terms equal to the square of the natural frequencies of free vibration (i.e., ω_j^2 $j = 1, \ldots, N$), and let $[U_f]$ represent the associated eigenvector matrix. Since $[M]$ and $[K]$ are both symmetric, then

$$[U_f^T][M][U_f] = [\Lambda_{mf}] \text{ and } [U_f^T][K][U_f] = [\Lambda_{kf}] \tag{6}$$

Also, Eq. (5) is recast in square matrix form[5] as

$$[K][U_f] = [M][U_f][\Lambda_\omega] \tag{7}$$

Combining Eqs. (6) and (7) gives

$$[\Lambda_{kf}] = [\Lambda_{mf}][\Lambda_\omega] \tag{8}$$

b. Static stability problem

If the inertia terms together with the vector $\{f(t)\}$ are neglected, Eq. (1) becomes

$$[[K] - p[P]]\{x\} = \{0\} \tag{9}$$

Let $[\Lambda_{cr}]$ be a diagonal matrix with terms equal to the critical buckling loads $(p_{cr})_j$ $j = 1, \ldots, N$ and let $[U_s]$ represent the associated eigenvector matrix. Since $[K]$ and $[P]$ are both symmetric, then

$$[U_s^T][K][U_s] = [\Lambda_{ks}] \text{ and } [U_s^T][P][U_s] = [\Lambda_p] \tag{10}$$

Equation (9) is rewritten in square matrix form as

$$[K][U_s] = [P][U_s][\Lambda_{cr}] \tag{11}$$

Combination of Eqs. (10) and (11) yields

$$[\Lambda_{ks}] = [\Lambda_p][\Lambda_{cr}] \tag{12}$$

For the condition of identical eigenvector matrices, it follows that

$$[U_s] = [U_f] \tag{13}$$

Referring to Eqs. (7) and (11), this condition holds[6] if and only if the matrix product $[M^{-1}][K]$ commutes with the matrix product $[P^{-1}][K]$. In simplified form the commutativity conditions takes the form

$$[P^{-1}][K][M^{-1}] = [M^{-1}][K][P^{-1}] \tag{14}$$

where $[M]$, $[K]$, and $[P]$ are assumed as nonsingular. In addition, the following matrix equality holds,

$$[\Lambda_{ks}] = [\Lambda_{kf}] \tag{15}$$

The general solution given in Eq. (2) takes a simplified form for this special case. Since the eignevectors are identical, it follows that

$$[U] = [U_s] = [U_f] \tag{16}$$

From Eqs. (3) and (6), one obtains

$$[\Lambda_m] = [\Lambda_{mf}] \tag{17}$$

Equation (3), together with Eqs. (10) and (12), yields

$$[\Lambda_{kp}] = [\Lambda_{ks}] - p[\Lambda_p] = [\Lambda_{ks}][[I] - p[\Lambda_{cr}^{-1}]] \tag{18}$$

Noting Eqs. (4, 8, 15, 17, and 18), it follows that

$$[\Lambda_\Omega] = [\Lambda_\omega][[I] - p[\Lambda_{cr}^{-1}]] \tag{19}$$

In scalar form the general jth equation of Eq. (19) is written as

$$\Omega_j^2 = \omega_j^2(1 - p/(p_{cr})_j) \quad j = 1, \ldots, N \tag{20}$$

where $0 < p < (p_{cr})_{min}$.

Thus, the natural frequency of free vibration of the system including axial forces is related to the natural frequencies of the system neglecting axial forces by Eq. (20) only for the special case where the eigenvector matrix simultaneously diagonalizes the $[M]$, $[K]$, and $[P]$ matrix individually.

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Exact Hydroelastic Solution for an Ideal Fluid in a Hemispherical Container

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DURING the past decade a variety of numerical hydroelastic analysis methods were developed for the study of fluid-structural dynamic interaction in launch vehicle propellant tanks. Most studies during this period were limited to axisymmetric hydroelastic dynamics. These results contributed to the analysis of coupled structural-propellant-engine system oscillations (POGO) of conventional axisymmetric launch vehicles. Recently, however, interest in asymmetric hydroelastic analysis methodology has been stimulated by the development of the Space Shuttle which is not dynamically axisymmetric.

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Verification of numerical hydroelastic methods is usually accomplished by comparison with experimental data and approximate analytical solutions but rarely by comparison with exact analytical solutions. In this Note, an, exact hydroelastic solution for free vibration of an incompressible irrotational fluid in an idealized hemispherical container is presented. Although the container idealization is unrealistic, asymmetric as well as axisymmetric vibration modes and natural frequencies are obtained which are believed to be quite useful in the verification of general hydroelastic analysis methods currently under development.

The hemispherical container under consideration follows the structural law $P = KU_r$ with P corresponding to the local fluid-structural interface pressure fluctuation, and U_r to the local outward normal displacement of the container wall. K is a structural stiffness constant.

The governing differential equations of small motion and boundary conditions consist of the continuity (Laplace) equation, the momentum equation, the fluid-structural interface boundary condition and the free surface boundary condition (neglecting gravitational surface waves). In terms of the spherical coordinate system defined in Fig. 1 and the displacement potential function, defined such that

$$\nabla\psi = U_r\hat{\imath}_r + U_\theta\hat{\imath}_\theta + U_\varphi\hat{\imath}_\varphi \tag{1}$$

these equations are, respectively

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} = 0 \tag{2}$$

$$P = -\rho\,\partial^2\psi/\partial t^2 \tag{3}$$

$$K\,\partial\psi/\partial r + \rho\,\partial^2\psi/\partial t^2 = 0 \text{ on } r = a,\ 0° \leqslant \theta \leqslant 90° \tag{4}$$

$$\psi, \text{ finite at } r = 0 \tag{5}$$

$$\psi = 0 \text{ on } \theta = \pi/2 \tag{6}$$

ρ is the fluid density. It should be noted that the boundary condition, Eq. (4), is a statement of the fluid-structural interaction arising from simultaneous enforcement of pressure and normal displacement compatability.

The exact solution of Eq. (2) subjected to Eq. (5) is in terms of uncoupled spherical harmonic functions

$$\psi_{MN} = (r/a)^M P_M{}^N(\cos\theta)\cos N\varphi\, e^{i\omega_{MN}t} \tag{7}$$

where $P_M{}^N(\cos\theta)$ is an associated Legendre function of degree M and order N.[1] The functions satisfying the free surface boundary condition, Eq. (6), consist of those with the index sum $M+N$ odd; the allowable functions are therefore M odd and N even or M even and N odd. In addition, the

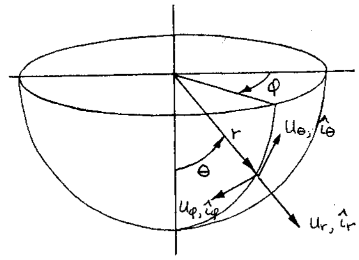

Fig. 1 Coordinate system.